Numerical Solution of Partial Differential Equations:

The general second-order linear partial differential equation is of the form:

$$A\frac{\partial^2 u}{\partial x^2}+B\frac{\partial^2 u}{\partial x\partial y}+C\frac{\partial^2 u}{\partial y^2}+D\frac{\partial u}{\partial x}+E\frac{\partial u}{\partial y}+Fu=G$$

Which can be written as:

$$Au_{xx}+Bu_{xy}+Cu_{yy}+Du_{x}+Eu_{y}+Fu=G \qquad ....(1)$$

Where $A, B, C, ..., G$ are all functions of x & y

$$au_{xx} + 2bu_{xy} + cu_{yy} = F(x, y, u, u_x, u_y)$$

u is the unknown function.

Equations of form (1) can be classified with respect to the sign of the discriminant:

$$\Delta s = B^2 - 4AC$$

in the following ways:

(1) If $\underline{\Delta s < 0}$ at a point in the (x, y) plane, the equation is said to be <u>elliptic type</u>.

(2) If $\underline{\Delta s > 0}$ at that point is said to be <u>hyperbolic type</u>.

(3) <u>Parabolic type</u> when $\underline{\Delta s = 0}$.

$$\text{Elliptic type} \quad 4ac - b^2 > 0 \quad \text{Laplace equation}$$

$$\text{Parabolic type} \quad 4ac - b^2 = 0 \quad \text{Heat equation}$$

$$\text{Heperbolic type} \quad 4ac - b^2 < 0 \quad \text{Wave equation}$$

In the following, we will restrict our solves to three simple particular cases of Eq. (1); namely:

$$u_{xx} + u_{yy} = 0 \qquad (\text{the Laplace equation})$$

$$u_{xx} - \frac{1}{c^2} u_{tt} = 0 \qquad (\text{the wave equation})$$

$$u_{xx} - u_t = 0 \qquad (\text{the heat conduction equation})$$

Finite – Difference Approximations to derivatives:

Let the (x, y) plane be divided into a network of rectangles of sides $\Delta x = h$ and $\Delta y = k$ by drawing the sets of lines:

$$x = ih; \qquad i = 0, 1, 2, 3, \ldots\ldots$$

$$y = jk; \qquad j = 0, 1, 2, 3, \ldots\ldots$$

The points of intersection of these families of lines are called mesh points, lattice points or grid points.

Then:

$$u_x = \frac{u_{i+1,j} - u_{i,j}}{k} + 0(h)\ Forward\ differenc$$

$$u_x = \frac{u_{i,j} - u_{i-1,j}}{h} + 0(h)\ Backward\ differenc$$

$$u_x = \frac{u_{i+1,j} - u_{i-1,j}}{2h} + 0(h^2) Centraldifferenc$$

$and;$

$$u_{xx} = \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2} + 0(h^2)$$

$$Where\ \ u_{i,j} = u(ih, jk) = u(x, y)$$

Similarly we have the approximations:

$$u_y = \frac{u_{i,j+1} - u_{i,j}}{k} + 0(k) \qquad \textit{Forward differenc}$$

$$u_y = \frac{u_{i,j} - u_{i,j-1}}{k} + 0(k) \qquad \textit{Backward differenc}$$

$$u_y = \frac{u_{i,j+1} - u_{i,j-1}}{2k} + 0(k^2) \qquad \textit{Central differenc}$$

and;

$$u_{yy} = \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{k^2} + 0(k^2)$$

We can now obtain the finite-difference analogues of partial differential equations by replacing the derivatives in any equation by their corresponding difference approximation given above.

Thus, the Laplace equation in two dimension, namely;

$$u_{xx} + u_{yy} = 0$$

has its finite-difference analogue;

$$\frac{1}{h^2}[u_{i+1,j} - 2u_{i,j} + u_{i-1,j}] + \frac{1}{k^2}[u_{i,j+1} - 2u_{i,j} + u_{i,j-1}] = 0$$

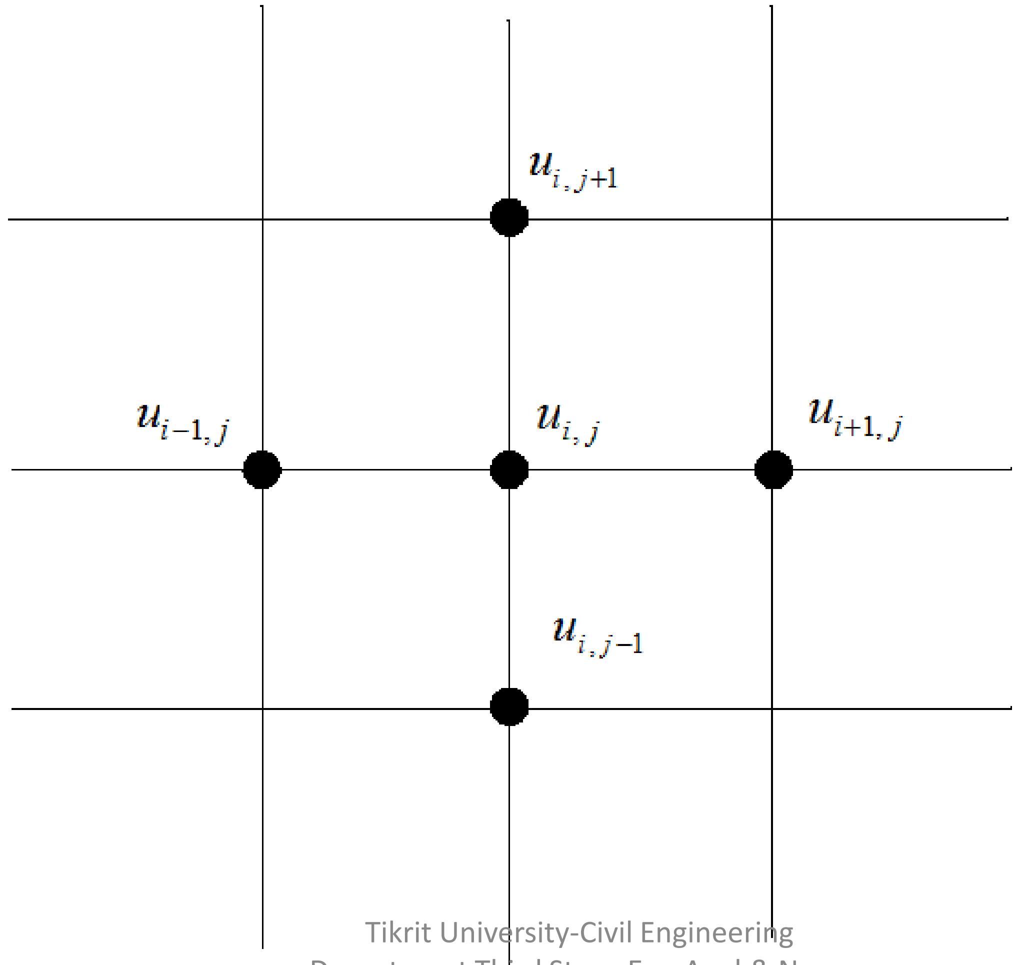
If $h = k$, *this gives;*

$$u_{i,j} = \frac{1}{4}[u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1}] \qquad (a)$$

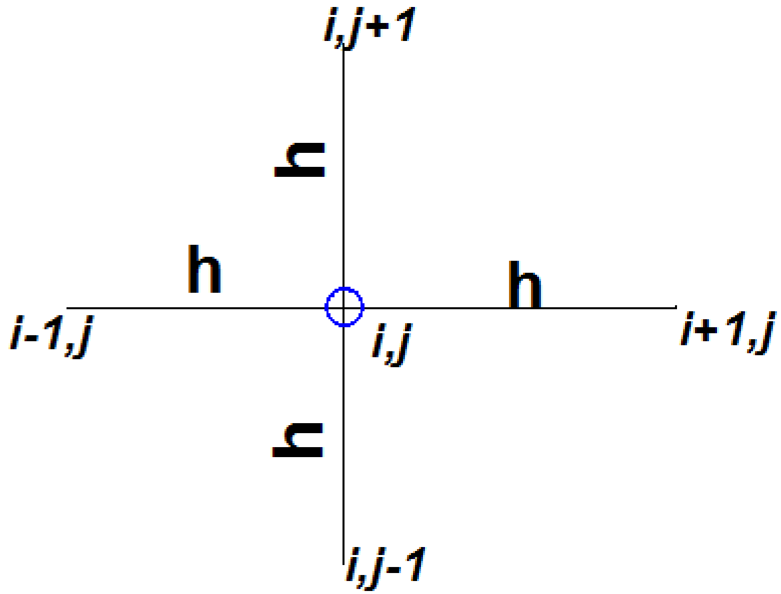
Which shows that the value of u at any point is the mean of its values at the <u>four neighbouring points</u>. This is called the <u>standard five-point formula</u>.

$The\ \underline{standard\ five-point\ formula}\ is\ written:$

$$u_{i+1,\,j} + u_{i-1,\,j} + u_{i,\,j+1} + u_{i,\,j-1} - 4u_{i,\,j} = 0$$



Dirichlet problem



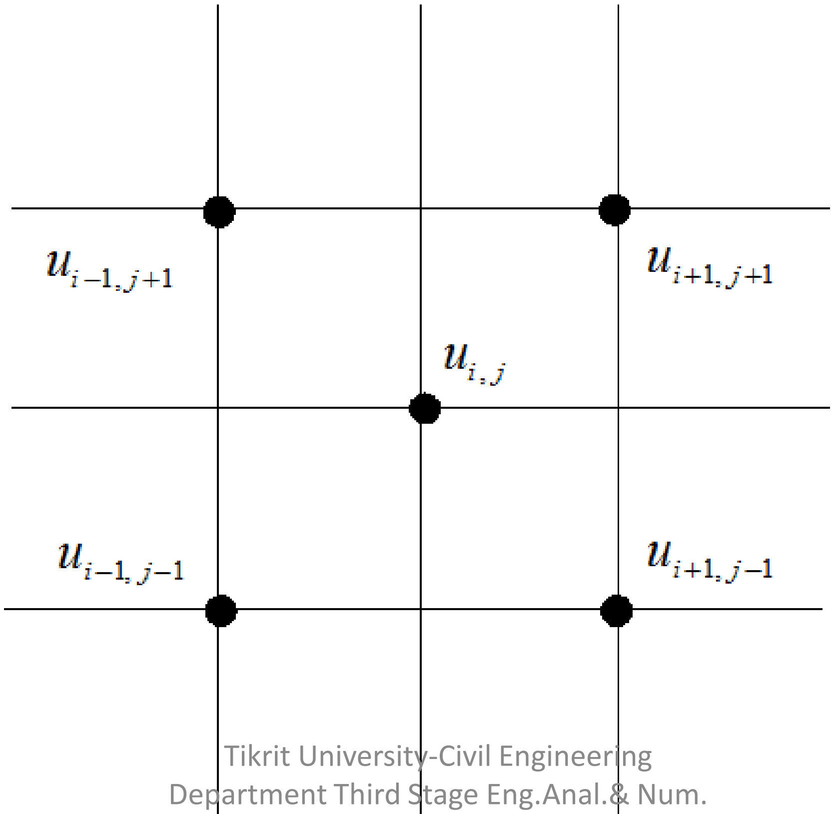
$$\begin{Bmatrix} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{Bmatrix}$$

Also, instead of formula (a), we may use the following formula:

$$u_{i,j} = \frac{1}{4}[u_{i-1,j-1} + u_{i+1,j-1} + u_{i+1,j+1} + u_{i-1,j-1}]$$

Which uses the function values at the <u>diagonal points</u>.

And is therefore called the <u>diagonal five-point formula</u>.

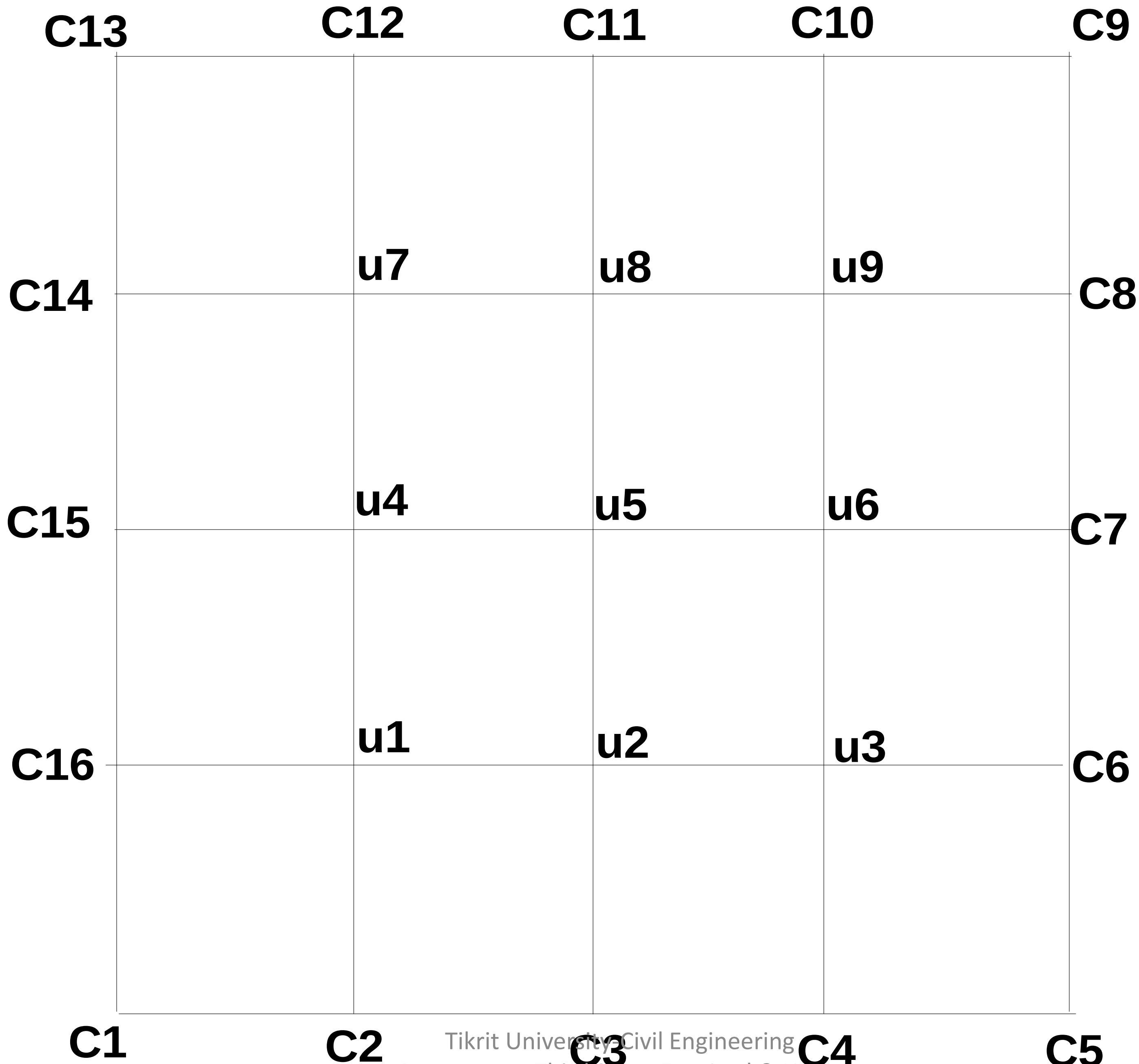


Laplace’s equation:

We wish to solve Laplace's equation:

$$u_{xx}+u_{yy}=0$$

in a boundary region *R* with boundary *C*. As in Dirichlet's problem, let the value of *u* be specified every where on *C*. For simplicity, let *R* be a square region so that it can be divided into a network of small squares of side *h*. Let the values of *u(x, y)* on the boundary *C* be given by *Ci* and let the interior mesh points and the boundary points be as in the figure below:


C13
C12
C11
C10
C9
u7
u8
u9
C14
C8
u4
u5
u6
C15
C7
u1
u2
u3
C16
C6
C1
C2
C3
C4
C5

We first use the diagonal five-point formula and compute; u_5, u_7, u_9, u_1 and u_3 in this order, Thus we obtain;

$$u_5 = \frac{1}{4}[C_1 + C_5 + C_9 + C_{13}];$$

$$u_7 = \frac{1}{4}[C_{15} + u_5 + C_{11} + C_{13}];$$

$$u_9 = \frac{1}{4}[u_5 + C_7 + C_9 + C_{11}];$$

$$u_1 = \frac{1}{4}[C_1 + C_3 + u_5 + C_{15}] \text{ and}$$

$$u_3 = \frac{1}{4}[C_3 + C_5 + C_7 + u_5]$$

We then compute, in this order, the remaining quantities, such u_8, u_4, u_6 *and* u_2 *by the standard five-point formula. Thus we have;*

$$u_8 = \frac{1}{4}[u_5 + u_9 + C_{11} + C_7];$$

$$u_4 = \frac{1}{4}[u_1 + u_5 + u_7 + C_{15}];$$

$$u_6 = \frac{1}{4}[u_5 + u_3 + C_7 + u_9] \text{ and}$$

$$u_2 = \frac{1}{4}[C_3 + u_3 + u_5 + u_1]$$

When once all the $u_i\,(i = 1,2,3,......9)$ are computed, their accuracy can be improved by any of the iterative methods described below:

(1) Jacobi's method:

Let $u_{i,j}^{(n)}$ denote the n^{th} iterative value of $u_{i,j}$.

An iterative procedure to solve the Eq.(a) is:

$$u_{i,j}^{(n+1)} = \frac{1}{4}[u_{i-1,j}^{(n)} + u_{i+1,j}^{(n)} + u_{i,j-1}^{(n)} + u_{i,j+1}^{(n)}]$$

for the interior mesh points. This is called the point Jacobi method.

(2) *Gauss – Seidal method:*

The method uses the latest iterative values available and scans the mesh points systematically from left to right along successive rows.

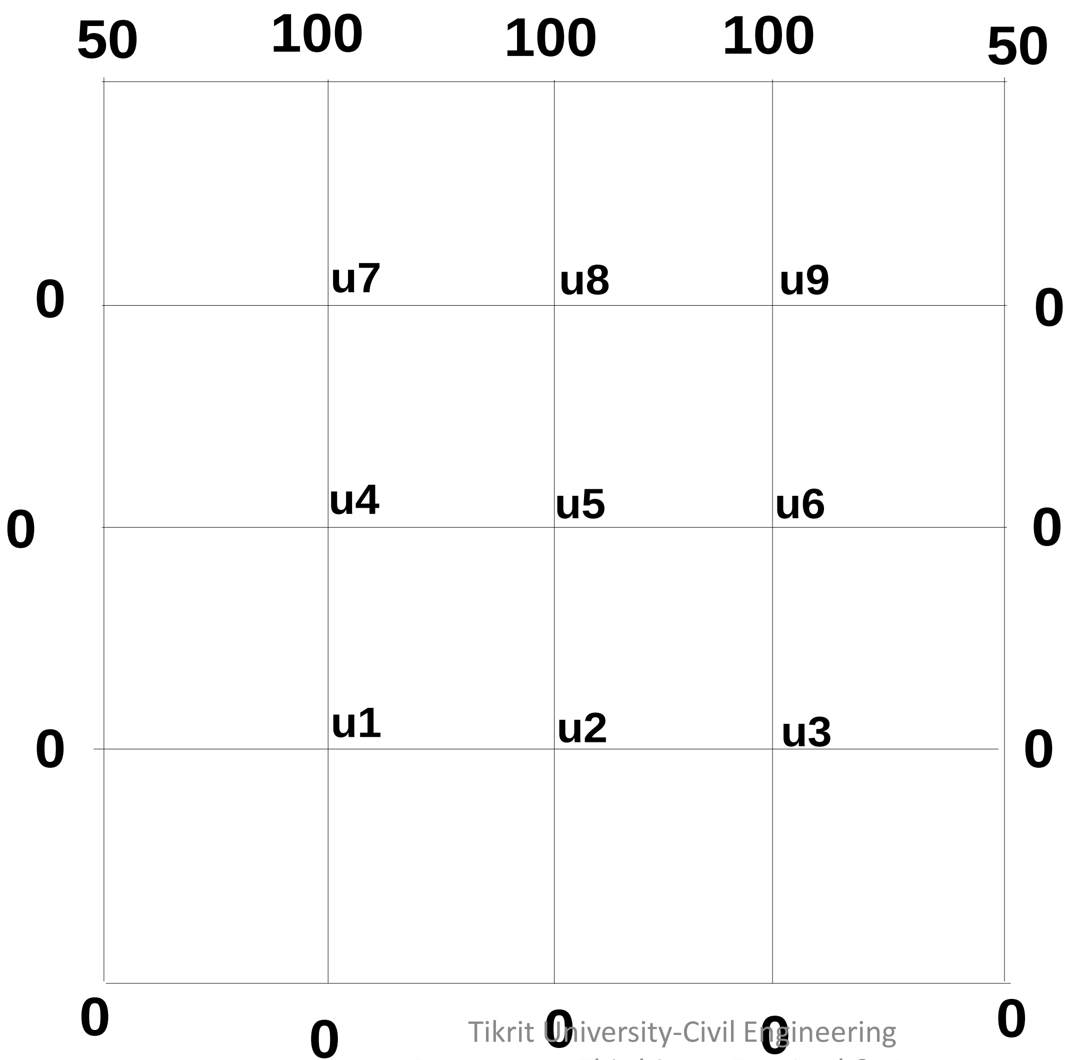
The iterative frmula is:

$$u_{i,j}^{(n+1)} = \frac{1}{4}[u_{i-1,j}^{(n+1)} + u_{i+1,j}^{(n)} + u_{i,j-1}^{(n+1)} + u_{i,j+1}^{(n)}]$$

It can be shown that the Gauss – Seidal scheme converges twice as fast as the Jacobi scheme.

Example:

Solve Laplace equation for the figure given below:



Solution:

We first compute the quantities u_5, u_7, u_9, u_1 and u_3 by using the diagonal five-point formula:

$$u_5^{(1)} = \frac{1}{4}[0+0+50+50] = 25$$

$$u_7^{(1)} = \frac{1}{4}[0+25+100+50] = 43.75$$

$$u_9^{(1)} = \frac{1}{4}[25+0+50+100] = 43.75$$

$$u_1^{(1)} = \frac{1}{4}[0+0+25+0] = 6.25$$

$$u_3^{(1)} = \frac{1}{4}[0+0+0+25] = 6.25$$

We now compute; u_8, u_4, u_6 and u_2 successively by using the standard five-point formula:

$$u_8^{(1)} = \frac{1}{4}[25 + 43.75 + 100 + 43.75] = 53.125$$

$$u_4^{(1)} = \frac{1}{4}[0 + 6.25 + 25 + 43.75] = 18.75$$

$$u_6^{(1)} = \frac{1}{4}[25 + 6.25 + 0 + 43.75] = 18.75$$

$$u_2^{(1)} = \frac{1}{4}[6.25 + 0 + 6.25 + 25] = 9.375$$

We have thus obtained the first approximations of all the nine mesh points and we can now use one of the aterative formula, by using Gauss − Seidal formula:

$$u_{i,j}^{(n+1)} = \frac{1}{4}[u_{i-1,j}^{(n+1)} + u_{i+1,j}^{(n)} + u_{i,j-1}^{(n+1)} + u_{i,j+1}^{(n)}]$$

n	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8	u_9
1	6.25	9.38	6.25	18.75	25.00	18.75	43.75	53.13	43.75
2	7.03	9.57	7.08	18.94	25.10	18.98	43.02	52.97	42.99
3	7.13	9.83	7.20	18.81	25.15	18.84	42.94	52.77	42.90
4	7.16	9.88	7.18	18.81	25.08	18.79	42.89	52.72	42.88
5	7.17	9.86	7.16	18.78	25.04	18.77	42.88	52.70	42.87

$$u_{1,1}^{(2)} = \frac{1}{4}[u_{0,1}^{(2)} + u_{2,1}^{(1)} + u_{1,0}^{(2)} + u_{1,2}^{(1)}]$$

$$u_1 = \frac{1}{4}[0 + 9.38 + 0 + 18.75] = 7.03$$

$$u_{2,1}^{(2)} = \frac{1}{4}[u_{1,1}^{(2)} + u_{3,1}^{(1)} + u_{2,0}^{(2)} + u_{2,2}^{(1)}]$$

$$u_2 = \frac{1}{4}[7.03 + 6.25 + 0 + 25] = 9.57$$